M321/E

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THE OPEN UNIVERSITY

Third Level Course Examination 1975

PARTIAL DIFFERENTIAL EQUATIONS OF APPLIED MATHEMATICS

Monday, 27th October, 1975 — 2.30 p.m. – 5.30 p.m.

Time allowed: 3 hours

You should attempt **NOT MORE THAN THREE** questions from Section A, and **NOT MORE THAN TWO** questions from Section B. Section A carries about 40% of the marks.

You may answer questions in any order, writing your answers in the answer book(s) provided. At the end of the examination, remember to write your name, student number and examination number on the answer book(s) – **failure to do so will mean that your paper(s) cannot be identified.**

Section A

Answer no more than THREE questions from this section. All questions in this section have equal marks.

Question 1

A good approximation to the equation governing the voltage $v(x, t)$ along a submarine cable is given by

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2},$$

where k is a positive constant and x, t represent distance along the cable and time respectively.

Suppose that the sending end of the cable ($x = 0$) is maintained at a constant positive voltage v_0 and that the receiving end ($x = l$) is earthed (i.e. maintained at zero voltage). Show that the steady-state voltage along the cable is $v_0(1 - x/l)$.

After this steady state has been established the sending end is earthed. This takes place at time $t = 0$. Determine the voltage $v(x, t)$ at subsequent times.

Question 2

The explicit finite-difference scheme

$$\frac{1}{k}\Delta_t u_{i,j} = \frac{1}{h^2}\delta_x^2 u_{i,j}$$

is used in approximating the problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial^2 U}{\partial x^2} && 0 < x < l,\ t > 0\ , \\ U(x, 0) &= f(x) && 0 \leqslant x \leqslant l\ , \\ U(0, t) &= g_1(t) && t > 0\ , \\ U(l, t) &= g_2(t) && t > 0\ . \end{aligned}$$

If the mesh ratio $r = k/h^2$ satisfies the inequality $r \leqslant \frac{1}{2}$, show that the scheme's solution satisfies the maximum principle

$$\max_{D_\Delta} u_{i,j} \leqslant \max_{C_\Delta} u_{i,j}\ ,$$

where, for any positive value of T, D_Δ is the set of mesh points interior to the solution domain with $t \leqslant T$ and C_Δ is the set of mesh points on the initial line and boundaries of the solution domain with $t \leqslant T$. Deduce from this that the scheme is stable for $r \leqslant \frac{1}{2}$.

What is the main practical disadvantage of this explicit method, and what method would you suggest might be preferable in practice?

Section A — *continued*

Question 3

Consider the problem

$$\begin{aligned} \nabla^2 u + ku &= F \quad \text{in } D \, , \\ u &= f \quad \text{on } C \, , \end{aligned}$$

where D is a plane region bounded by the closed curve C, k is a nonzero constant, and F, f are functions defined on D, C respectively.

Use Green's theorem to show that if $k < 0$ any solution to the problem is unique.

Suppose that

$$D = \{(x, y) : 0 < x < a, 0 < y < b\} \, ,$$

and that

$$\frac{k}{\pi^2} = \frac{m^2}{a^2} + \frac{n^2}{b^2}$$

for some pair of nonzero integers m, n. By finding a solution to the corresponding homogeneous problem, or otherwise, show that in this case the original problem cannot have a unique solution.

Question 4

Find the Green's function for the system

$$\begin{aligned} u'' + u &= -f \qquad 0 < x < \pi \, , \\ u'(0) - u(0) &= u(\pi) = 0 \, . \end{aligned}$$

Hence write down an expression for the solution to the problem and verify that this satisfies the given boundary conditions.

Section B

Answer no more than TWO questions from this section. All questions in this section have equal marks.

Question 5

Verify that the operator

$$L \equiv \frac{\partial^2}{\partial t^2} + 2x\frac{\partial^2}{\partial x \partial t} + (x^2 - 4)\frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial t} + (3x + 2)\frac{\partial}{\partial x}$$

is hyperbolic. Show that its characteristic coordinates may be defined by

$$\xi = (x - 2)\, e^{-t} ,$$
$$\eta = (x + 2)\, e^{-t} ,$$

and hence derive the standard (canonical) form of the equation $L[u] = 0$ as

$$(\xi - \eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial u}{\partial \xi} = 0 .$$

Find the general solution of this last equation in the form

$$u(\xi, \eta) = f(\xi) + g(\eta) + (\eta - \xi)f'(\xi) ,$$

where f, g are suitably differentiable but otherwise arbitrary functions.

Deduce that the solution of the problem

$$L[u] = 0 \qquad x \in R,\ t > 0 ,$$
$$u(x, 0) = p(x) \qquad x \in R ,$$
$$\frac{\partial u}{\partial t}(x, 0) = (2 - x)p'(x) \qquad x \in R ,$$

is

$$u(x, t) = e^{-t} p(w_+) + \tfrac{1}{4}\int_{w_-}^{w_+} p(z)\, dz ,$$

where

$$w_\pm = x\, e^{-t} \pm 2(1 - e^{-t}) .$$

Section B — *continued*

Question 6

Show that the finite-difference scheme

$$\frac{1}{k}\,\Delta_t\{(1+\alpha\delta_x^2)u_{i,j}\} = \frac{1}{h^2}\,\delta_x^2 u_{i,j}$$

approximating the problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad 0 < x < 1,\ t > 0\ ,$$

$$U(0, t) = U(1, t) = 0 \qquad t > 0\ ,$$

$$U(x, 0) = f(x) \qquad 0 \leqslant x \leqslant 1\ ,$$

is unconditionally unstable for $\alpha \geqslant \frac{1}{4}\sec^2\frac{1}{2}\pi h$, and stable if

$$2\alpha + r \leqslant \tfrac{1}{2}\ ,$$

where $r = k/h^2$ is the mesh ratio. For what value of α is the scheme explicit? Show that the value $\alpha = -\frac{1}{2}r$ gives the Crank-Nicolson scheme for the problem. What can you deduce about the stability of each of these schemes?

Use the Taylor series expansions

$$\delta_x^2 U\,(x, t) = 2\left(\frac{h^2}{2!}\frac{\partial^2}{\partial x^2} + \frac{h^4}{4!}\frac{\partial^4}{\partial x^4}\right) U\,(x, t) + O(h^6)\ ,$$

$$\Delta_t P\,(x, t) = \left(k\frac{\partial}{\partial t} + \frac{k^2}{2!}\frac{\partial^2}{\partial t^2}\right) P\,(x, t) + O(k^3)\ ,$$

to show that when α is independent of r the finite-difference scheme has a local truncation error of order $O(k) + O(h^2)$. Show that the Crank–Nicolson scheme has a local truncation error of order $O(k^2) + O(h^2)$. Find also a value of α for which the scheme has a local truncation error of order $O(k^2) + O(kh^2) + O(h^4)$.

Section B — *continued*

Question 7

A square ice rink of side l is bounded by a pipe maintained at a steady temperature u_b°C, where $u_b < 0$. The rink is insulated beneath, but gains heat from the atmosphere above it which remains at a constant temperature u_a°C, where $u_a > 0$.

If the ice is sufficiently thin, the rink may be adequately represented by the region $[0, l] \times [0, l]$ of R^2. By considering the heat balance in a small square of the rink bounded by the lines $x = x_0$, $x = x_0 + \Delta x$, $y = y_0$, $y = y_0 + \Delta y$, or otherwise, show that the steady-state temperature $u(x, y)$ of the rink may be described by the equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + h^2(u_a - u) = 0 \qquad 0 < x < l, 0 < y < l\,,$$

$$u(x, 0) = u(x, l) = u_b \qquad 0 \leqslant x \leqslant l\,,$$

$$u(0, y) = u(l, y) = u_b \qquad 0 \leqslant y \leqslant l\,,$$

where h is a real constant. You may assume

(*a*) that the rink gains heat from the atmosphere at a rate per unit area proportional to the temperature difference between the rink and the air above it;

(*b*) that the rate of flow of heat per unit area across any surface within the ice is proportional and of opposite sign to the temperature derivative normal to that surface.

Using the finite Fourier transform method, show that the temperature distribution of the rink can be expressed as

$$u(x, y) = u_b + \frac{4l^2h^2}{\pi}(u_a - u_b) \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{np_n^2}\left\{1 - \frac{\cosh p_n\left(\frac{y}{l} - \frac{1}{2}\right)}{\cosh \frac{1}{2}p_n}\right\} \sin\frac{n\pi x}{l}\,,$$

where $p_n = \sqrt{n^2\pi^2 + l^2h^2}$. Derive a condition independent of x and y sufficient to ensure that the entire rink is in fact frozen.

Section B — *continued*

Question 8

Let $v(x)$ be any solution of the equation

$$v'' + \lambda \rho v = 0 \qquad x > 0 ,$$

where ρ is a monotonically increasing positive function of x and λ is a positive constant. Let α_n be the nth non-negative zero of v. By defining new coordinates

$$x_i = x - \alpha_i \qquad i = n - 1, n ,$$

and corresponding functions

$$\rho_i(x_i) = \rho(x) ,$$
$$v_i(x_i) = v(x) ,$$

use the Monotonicity Theorem to show that

$$\alpha_{n+1} - \alpha_n < \alpha_n - \alpha_{n-1} \quad . \ n = 2, 3, \ldots .$$

Let λ_2 be the second eigenvalue of the system

$$u'' + \lambda(1 + x)u = 0 \qquad 0 < x < 1 ,$$
$$u(0) = u(1) = 0 .$$

By comparing the system with others having constant coefficients, show that

$$2\pi^2 < \lambda_2 < 4\pi^2 .$$

It is known from the Oscillation Theorem that the second eigenfunction of the system above has precisely one zero (say α) in the interval (0, 1). By considering the problem

$$u'' + \lambda(1 + x)u = 0 \qquad 0 < x < \alpha ,$$
$$u(0) = u(\alpha) = 0 ,$$

prove that

$$\tfrac{1}{2} < \alpha < \sqrt{\tfrac{1}{2}} .$$

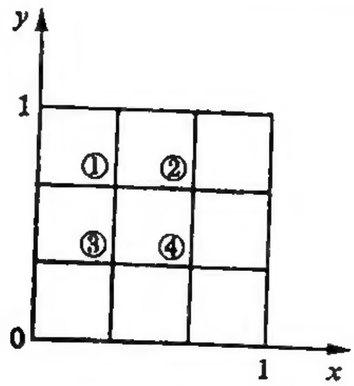
Section B — *continued*

Question 9

Write down the five-point finite-difference approximation for the problem

$$\begin{aligned} \nabla^2 U(x, y) &= f(x, y) & 0 < x < 1, 0 < y < 1\ , \\ U(x, 0) = U(x, 1) &= 0 & 0 \leqslant x \leqslant 1\ , \\ U(0, y) = U(1, y) &= 0 & 0 \leqslant y \leqslant 1\ , \end{aligned}$$

where the mesh length is taken to be $\frac{1}{3}$ in each direction.



Show that when the finite-difference equations at the points 1, 2, 3, 4 (see diagram above) are taken in their natural order, the resulting matrix of coefficients is consistent. Give also an ordering of the equations whose matrix of coefficients has the block form

$$\begin{bmatrix} D_1 & F \\ E & D_2 \end{bmatrix},$$

where D_1, D_2 are diagonal matrices. Is this ordering consistent with the previous one? How will the asymptotic rates of convergence for the two orderings compare if a solution to the equations is sought using the SOR iterative method?

Given that the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

are 0, 0, ± 2, show using the natural ordering of the finite-difference equations that the optimum value of the SOR parameter ω is $4(2 - \sqrt{3})$. Show also that if this value of ω is used in the SOR method, the error of the approximation will ultimately decrease by a factor of about 14 at each successive step of the iteration. [$\sqrt{3} \simeq 1.73$]

Section B — *continued*

Question 10

Show that for $m \geqslant 0$ and $\lambda > 0$,

$$\int_0^1 [J_m(\lambda x)]^2 x\, dx = \tfrac{1}{2}[J_m'(\lambda)]^2 + \frac{\lambda^2 - m^2}{2\lambda^2}[J_m(\lambda)]^2 ,$$

stating without proof any more basic identities involving Bessel functions that you use in the process (J_m is the Bessel function of the first kind of order m).

The equations

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad 0 < r < 1, 0 < z < a ,$$

$$\frac{\partial u}{\partial r}(1, z) = 0 \qquad 0 \leqslant z \leqslant a ,$$

$$u(r, a) = 0 \qquad 0 < r < 1 ,$$

$$u(r, 0) = 1 - r^2 \qquad 0 < r < 1 ,$$

describe the steady-state temperature $u(r, z)$ in a cylindrical region of length a and radius 1 which is insulated along its length and has certain radially symmetric temperature distributions maintained at each end. Given that the set of functions

$$\{1, J_0(\mu_k r) : \mu_k = \text{the } k\text{th positive root of the equation } J_0'(\mu) = 0, k = 1, 2, \ldots\}$$

is complete on the interval $[0, 1]$ with respect to the weight function r, show that the solution of the problem may formally be expressed as

$$u(r, z) = \frac{a - z}{2a} + 4\sum_{k=1}^{\infty} \frac{J_2(\mu_k)}{\mu_k^2[J_0(\mu_k)]^2}\,\frac{\sinh \mu_k(a - z)}{\sinh \mu_k a}\,J_0(\mu_k r) .$$